

# Is classical reality completely deterministic?

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## Abstract

The concept of determinism for a classical system is interpreted as the requirement that the solution to the Cauchy problem for the equations of motion governing this system be unique. This requirement is generally assumed to hold for all autonomous classical systems. We give counterexamples of this view. Our analysis of classical electrodynamics in a world with one temporal and one spatial dimension shows that the solution to the Cauchy problem with the initial conditions of a particular type is not unique. Therefore, random behavior of closed classical systems is indeed possible. This finding provides a qualitative explanation of how classical strings can split. We propose a modified path integral formulation of classical mechanics to include indeterministic systems.

## 1 Introduction

According to present views, the quantum is fundamentally random. By this is meant that quantum mechanics is a probabilistic theory and there are no deterministic laws underlying quantum phenomena. By contrast, the classical is regarded as deterministic. Of course, classical statistical mechanics invokes probability theory, but the reason for this is different from that of quantum mechanics. Uncertainties in classical statistical mechanics may be attributed to lack of knowledge of actual deterministic histories of macroscopic systems which have too many degrees of freedom to be completely controlled.

Worthy of mention are also classical stochastic systems (among which are systems with some few degrees of freedom) [1]. Although stochastic mechanics is formulated with the help of probability theory, stochasticity should not be confused with randomness. Classical stochastic systems are governed by deterministic laws. The gist of the question is that their histories are depicted by tangled trajectories. Motions displaying extreme sensitivity to initial conditions are commonly viewed as stochastic. Complexity effects in the behavior of unstable systems are a major manifestation of stochasticity. To be more exact, a system is defined as stochastic if there is a compact region confining the motion  $x(t)$  in which  $x(t)$  depends heavily on initial data  $x_0$ :

$$\frac{\partial x(t; x_0)}{\partial x_0} \sim \exp(t/\Delta), \quad t \gg \Delta \quad (1)$$

(where  $\Delta$  stands for a characteristic time interval). The apparent indeterminism in the behavior of stochastic systems is then fictitious; it is due to imperfect knowledge of initial

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conditions. We can in principle specify  $x_0$  with arbitrary accuracy, and thereby predict the history  $x(t; x_0)$  as precisely as desired.

To discern phenomena which indeed run counter to Laplace's determinism, we must refine upon this paradigm. We say that Laplace's determinism holds for a given system if the Cauchy problem for the equations of motion governing this system—whenever the initial conditions—has a *unique* solution. This requirement is generally believed to be imperative in classical physics. Strange as it may seem, there are autonomous classical systems in two-dimensional spacetime which violate this principle. Examples of such indeterministic systems are given below. We will see that behavior of these systems must be recognized as truly random. In two-dimensional worlds, God does roll the dice.

It may be that this implication will have some utility in string theory. By now, there has been remained an open question of whether fundamental strings can split on the classical level. At first sight, classical strings are unable to split at all. Take, for example, an open Nambu string. It can be indefinitely stretched without no evidence of being favorably disposed towards splitting<sup>2</sup>: there is no elastic limit for objects governed by the Nambu action. Indeed, the only dimensional parameter in this action is  $1/2\pi\alpha'$  which is merely an overall factor that defines the scale of length. It follows that classical strings are immune from compulsory splittings. However, as will transpire in Sect. 3, *spontaneous* splittings are yet feasible in the classical picture.

The paper is organized as follows. In Sect. 2 we explore a particle on the top of a hill. From this discussion, a general idea can be had of how classical systems can reveal its indeterministic nature. Classical electrodynamics of point particles in a world with one temporal and one spatial dimension is analyzed in Sect. 3. We show that exact solutions to the Cauchy problem for the set of dynamical equations governing a closed system of two charged particles and the electromagnetic field can be not unique. We then propose a toy model which qualitatively explains random splitting of classical strings. In the final section, we turn to the path integral formulation of classical mechanics [3]–[5]. If indeterministic systems are to be incorporated, the classical path integral construction should be properly modified. We outline a possible way for this modification.

## 2 At the top of a hill

Let us take a closer look at two like charged particles which move towards each other along a straight line. Having spent kinetic energy for overcoming the interparticle repulsion by their meeting these particles merge into a single point aggregate. Since our concern is with final stage of this head-on collision when velocities of the particles are close to zero, the use of nonrelativistic approximation would be quite accurate.

The two-particle problem can be brought to a one-particle problem if we introduce

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<sup>2</sup>See, however, Ref. [2] where a thermodynamical argument in support of the idea that some classical string configurations show tendency to split is adduced. It seems appropriate to reason that some splitted configuration is more advantageous in mass content than its associated unbroken configuration, but this criterion is in general insufficient to determine the point of the string where splitting actually occurs.

the relative coordinate  $r = x_2 - x_1$ , reduced mass  $m = m_1 m_2 / (m_1 + m_2)$ , and potential energy  $U(r)$ . The problem is then to describe a particle climbing to the hill  $U(r)$  so that its velocity vanishes on its arrival at the top of the hill, see Fig. 1. Let the coordinate of

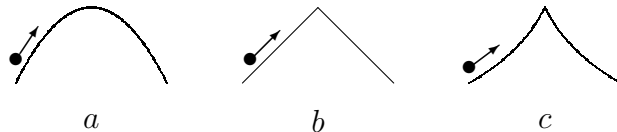


Figure 1: Ascent to the top of a hill

the top be  $r = 0$ ,  $U_{\max} = U(0)$ . The time that the particle comes to the top is chosen to be  $t = 0$ . Vanishing the particle's velocity at  $r = 0$  means that the total energy is zero,

$$E = \frac{1}{2} m \dot{r}^2 + U(r) = 0. \quad (2)$$

The time it takes for the particle to arrive at the top is therefore

$$t(r) = -\sqrt{\frac{m}{2}} \int_r^0 \frac{dx}{\sqrt{-U(x)}}. \quad (3)$$

What happens to the particle upon its arrival at the top? (Although this question seems sound, no attempt has been made of answering it. The only exception is Ref. [6], where this issue is addressed, but its solution is left to the reader.) If the integral in (3) diverges, then the question of the subsequent lot of this particle does not arise because the ascent takes infinite time. Such is indeed the case for  $U(r)$  which is an analytical function at  $r = 0$ , say, for  $U(r) = -U_0 r^2$ . Meanwhile the integral is finite for

$$U(r) = -U_0 r^{2(1-\alpha)}, \quad 0 < \alpha < 1, \quad (4)$$

$$U(r) = -U_0 r^2 (\ln r^2)^{2(1+\beta)}, \quad \beta > 0, \quad (5)$$

$$U(r) = -U_0 r^2 (\ln r^2)^2 [\ln (\ln r^2)^2]^{2(1+\gamma)}, \quad \gamma > 0, \quad (6)$$

and the like.

The differential equation (2) is invariant under time reversal. Furthermore,  $r(t) = 0$  is another solution to (2). Therefore, if the climb takes a finite period of time, then a continual set of options is available: after staying at the top for an arbitrary period of time  $T$ , the particle can start to descend in either direction<sup>3</sup>. Analytically,

$$r(t) = \begin{cases} f(t) & t < 0, \\ 0 & 0 \leq t < T, \\ \pm f(T - t) & t \geq T, \end{cases} \quad (7)$$

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<sup>3</sup>If we bear in mind the initial two-particle problem and assume that colliding particles are unable to penetrate through each other, then both ascent and descent in the effective one-particle problem are described by positive values of  $r$ .

where  $f(t)$  is the inverse of  $t(r)$  defined in (3).

By the Picard theorem, the solution to the Cauchy problem for the differential equation (2) with the initial condition  $r = 0$  is unique if the Lipschitz condition holds,

$$\sqrt{-U(r)} < C|r|. \quad (8)$$

Here,  $C$  is some positive constant. Clearly, for  $U(r)$  given by (4)–(6), inequality (8) fails.

We thus infer that potentials, which are visualized as hills, are divided into two classes: unstable potentials of the conventional type and *over-unstable* potentials. The equilibrium state in potentials of the conventional type is kept until a small external perturbation occurs, whereas this state in over-unstable potentials can be upset *spontaneously*, that is, with no external cause.

The Lipschitz condition is sufficient but not necessary for stability against spontaneous decays. Convergence of the integral in (3) may serve a necessary condition. For example, inequality (8) does not hold for  $U = -U_0 r^2 (\ln r^2)^2$ , even if the integral in (3) diverges. Note also the absence of a strict analytical demarcation line between unstable potentials of the conventional type and over-unstable potentials, in particular, the sequence of over-unstable potentials shown in (4)–(6) extends indefinitely.

A striking thing is that a particle at rest shows the capacity for sliding down the hill without any causation and thus at random. This phenomenon is in conflict with Laplace's determinism. Going back to the initial two-particle problem, we see that the aggregate of two merged particles will spontaneously disintegrate into its constituents after a lapse of an arbitrary interval  $T$ . Note that  $T = \infty$  is among possible options, that is to say the aggregate can remain fixed for an infinitely long time.

The sceptical reader may disregard these issues for several compelling reasons. First, head-on collisions of point particles are highly improbable on a three-dimensional arena: the probability measure of such events is zero. Second, the interaction potential  $U(r)$  like that shown in (4)–(6) seems to have little (if any) significance as an element of physical reality. Third, time reversal is crucial for spontaneous equilibrium breaking to occur. Once accelerated charges radiate electromagnetic energy, the dynamics becomes dissipative and irreversible, and hence solution (7) ceases to be true. Fourth, to ensure that two colliding particles amalgamate in a single aggregate, their total energy must be exactly zero. The initial data of the corresponding Cauchy problems constitute a null set.

All these objections can be withdrawn if we turn to a world with one temporal and one spatial dimension. First, observe that, for particles living in a line, head-on collisions are not uncommon. Second, with reference to [7, 8], we recall that the time component of the retarded vector potential  $A_\mu$  in two-dimensional electrodynamics is given by  $A_0 = -e|r|$  which, on putting  $\alpha = \frac{1}{2}$ , falls into the type of (4), Fig. 1b. Third, it was shown in [7, 8] that charged particles in two-dimensional spacetime do not radiate. Therefore, all processes in this realm are reversible. Fourth, although the case that the total energy of two colliding particles is zero is indeed extremely exotic, it is possible to customize the very problem setting with a tangible ground. Let a particle be capable of spontaneous decaying into two interacting particles, with the total energy of this system being equal to zero. Then one extends analytically this history back in time according to Eq. (7). Of

course, this trick only helps in rendering the “real” history of forming the aggregate of two particles a virtual history (which is to drop out of sight). Hence, it may be argued that letting the existence of such point aggregates does not stand up. The key step is to switch from particular aggregates to a continual set of identical aggregates constituting a string. Leaving aside the origin of such sets, we take advantage of discrete toy models of a string for better understanding the classical mechanism of its splitting.

### 3 Two-dimensional world

We now consider classical electrodynamics in two-dimensional spacetime. Our notations are identical to those of Ref. [8]. The action for a system of  $N$  charged point particles and the electromagnetic field is given by

$$S = - \sum_{I=1}^N m_I \int ds_I \sqrt{\dot{z}_I^\mu \dot{z}_I^\mu} - \int d^2x \left( \frac{1}{8} F_{\mu\nu} F^{\mu\nu} + A^\mu j_\mu \right), \quad (9)$$

$$j^\mu(x) = \sum_{I=1}^N e_I \int_{-\infty}^{\infty} ds_I \dot{z}_I^\mu(s_I) \delta^{(2)}[x - z_I(s_I)]. \quad (10)$$

Here, the field strength  $F_{\mu\nu}$  is related to the vector potential  $A_\mu$  in the conventional way

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (11)$$

Varying  $A^\mu$  and  $z_I^\mu$ , we have

$$\partial_\lambda F^{\lambda\mu} = 2j^\mu, \quad (12)$$

$$m_I \ddot{z}_I^\mu = e_I \dot{z}_I^\alpha F^{\mu\alpha}(z_I). \quad (13)$$

A remarkable fact is that this system of equations is completely integrable [8]. The procedure of finding solutions to this system is rather standard. First we obtain a retarded solution to the field equation (12) with the source composed of  $N$  charges moving along arbitrary smooth timelike world lines. The notation  $R_I^\mu = x^\mu - z_I^\mu(s_I^{\text{ret}})$  is used to denote the null vector drawn from the emission point  $z_I^\mu(s_I^{\text{ret}})$  on the  $I$ th world line to the point of observation  $x^\mu$ . From here on the mark ‘ret’ will be suppressed. We introduce a further null vector  $c_I^\mu$  related to  $R_I^\mu$  by

$$R_I^\mu = \rho_I c_I^\mu, \quad (14)$$

where

$$\rho_I = \dot{z}_I \cdot R_I \quad (15)$$

is the distance between emission and observation points in the frame in which the time axis is aligned with  $\dot{z}_I^\mu$ . The retarded solution to (11)–(12) can be written [8] as

$$A^\mu = - \sum_{I=1}^N e_I R_I^\mu, \quad (16)$$

$$F^{\mu\nu} = \sum_{I=1}^N e_I (c_I^\mu \dot{z}_I^\nu - c_I^\nu \dot{z}_I^\mu). \quad (17)$$

As an illustration let us consider the case  $N = 2$ . This two-particle problem can be translated into the problem of motion of two parallel plates of a planar immense capacitor. Evidently there is only an electric field  $\mathbf{E}$  between the plates, which is constant for any separation and velocity of the plates.

Applying (17) to the symmetric stress-energy tensor of the electromagnetic field

$$\Theta^{\mu\nu} = \frac{1}{2} \left( F^{\mu\alpha} F_\alpha{}^\nu + \frac{\eta^{\mu\nu}}{4} F_{\alpha\beta} F^{\alpha\beta} \right) \quad (18)$$

gives  $\Theta_{\text{self}}^{\mu\nu} + \Theta_{\text{mix}}^{\mu\nu}$  where  $\Theta_{\text{self}}^{\mu\nu}$  is the sum of terms each containing only the field generated by a particular charge, and  $\Theta_{\text{mix}}^{\mu\nu}$  contains mixed contributions. Let  $\Theta_I^{\mu\nu}$  be a term of  $\Theta_{\text{self}}^{\mu\nu}$  due to the  $I$ th charge. We have

$$\Theta_I^{\mu\nu} = \frac{1}{4} e_I^2 \eta^{\mu\nu}. \quad (19)$$

This expression suggests that there is no radiation in two-dimensional spacetime (for an extended discussion of this subject see [7, 8].)

If we substitute (17) in (13) and solve the resulting equations, then we find that every particle moves along a hyperbolic world line [8]. In “degenerate” cases, the history of a particle is represented by straight world lines.

We now return to the system of two colliding particles, discussed in the previous section. For simplicity, we choose the barycentric frame, and assume that the particles have equal masses  $m$  and charges  $e$ . Our concern here is with the case that velocities of the particles are precisely zero at the instant of their meeting. The exact solution [9] is represented by two world lines  $z_1^\mu(s)$  and  $z_2^\mu(s)$  which coalesce at  $s = s^*$  and separate at  $s = s^{**} = s^* + T$ ,

$$z_1^\mu(s) = \begin{cases} a^{-1} (\sinh a(s - s^*), 1 - \cosh a(s - s^*)) & s < s^*, \\ (s - s^*, 0) & s^* \leq s < s^{**}, \\ a^{-1} (aT + \sinh a(s - s^{**}), \cosh a(s - s^{**}) - 1) & s \geq s^{**} \end{cases}, \quad (20)$$

$$z_2^\mu(s) = \begin{cases} a^{-1} (\sinh a(s - s^*), \cosh a(s - s^*) - 1) & s < s^*, \\ z_1^\mu(s) & s^* \leq s < s^{**}, \\ a^{-1} (aT + \sinh a(s - s^{**}), 1 - \cosh a(s - s^{**})) & s \geq s^{**}. \end{cases} \quad (21)$$

Here,  $a = e^2/m$ .

The parameters  $s^*$  and  $s^{**}$  are arbitrary. If  $s^*$  and  $s^{**}$  are different and finite, then Eqs. (20) and (21) correspond to the history of an aggregate with finite life time. If  $s^{**} \rightarrow \infty$ , then this solution represents the history of a stable aggregate originated at a finite instant. In the limit  $s^* \rightarrow -\infty$ , we have the history of an aggregate, formed at the infinitely remote past, whose decay occurs at a finite instant. If  $s^* \rightarrow -\infty$  and  $s^{**} \rightarrow \infty$ , then this solution becomes a straight line corresponding to an absolutely stable aggregate. For  $s^* = s^{**}$ , this solution describes an aggregate existing for a single moment.

We thus see that the exact solution to the Cauchy problem for the set of equations governing a closed system of two charged particles and the electromagnetic field in two-dimensional spacetime, with the initial condition that the total energy of this system is zero, is not unique. In fact, we have a continuum of solutions (20)–(21) where  $T$  is arbitrary: the aggregate disintegrates quite accidentally at any instant after its formation<sup>4</sup>.

To apply this result to strings, we think of them as chain structures. For example, a system of two particles, which are held together by the linearly rising potential (16), resembles an open string whose energy is linear in its length. While the particles exchange electromagnetic signals along the two-dimensional light cone, string perturbations (in the orthonormal gauge) are governed by the wave equation  $X_{\tau\tau} - X_{\sigma\sigma} = 0$  whose characteristic surface is the light cone in the  $(\tau, \sigma)$ -plane. If a two-parameter family of curves, labelled by  $\tau$  and  $\sigma$ , is drawn perpendicular to the world lines of the particles, then we have a toy discrete model of strings with Dirichlet boundary conditions  $X(\tau, 0) = X(\tau, l) = 0$ .

$N$ -particle clusters with  $N > 2$  are also suitable for modeling such strings. It is possible to follow the course of joining of two open strings into one and subsequent spontaneous splitting of this string into two pieces if the extreme left particle of a cluster on the right and the extreme right particle of a cluster on the left move to meet (Fig. 2a) and merge into a single point aggregate (Fig. 2b), and then, after a lapse of a time interval  $T$ , this aggregate disintegrates into two initial particles (Fig. 2c), according to Eqs. (20)–(21). One may then deem a classical string to be a set of aggregates of this kind. Spontaneous disintegration of some element of this set is the reason for splitting of the string.

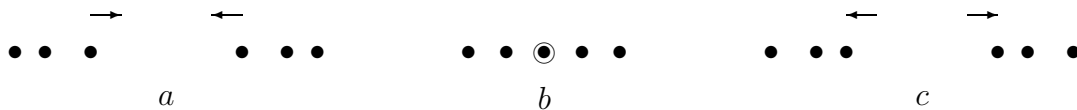


Figure 2: Joining and splitting of “strings”

One may wish to use the Yang–Mills theory with point particles endowed with color charges transforming as the adjoint representation of the gauge group in constructing such string models as an alternative to electrodynamics. But this analysis accomplishes nothing new: all retarded solutions to the Yang–Mills equations in two-dimensional spacetime are Abelian, that is, they can be built with the aid of the Cartan subgroup of the gauge group [7], so that we revert to the situation in two-dimensional electrodynamics.

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<sup>4</sup>It may be worth noticing once again that the colliding particles cannot bounce off in the ordinary way because both velocities and interparticle repulsion vanish at the instant of their meeting  $s = s^*$ .

## 4 Refinement of the path-integral concept in classical mechanics

The path-integral formulation of classical mechanics developed in [3]–[5] opened up new avenues for studies of the connection between the quantum and the classical. A central idea of this approach is that the classical path integral is contributed by a single path that renders the action extremal. Useful though this concept for a large class of truly deterministic systems, it must be modified if we wish to incorporate systems violating Laplace’s determinism. We now discuss a possible way for this modification. But before proceeding further, we make a cautionary remark on a troublesome aspect of this concept.

It is common for the path-integral approach to take the principle of least action in Hamilton’s form

$$S = \int_0^T dt L(q, \dot{q}), \quad \text{or} \quad S = \int_0^T dt [p\dot{q} - H(q, p)]. \quad (22)$$

One may inquire: what is the classical transition amplitude  $K(\phi_f, T|\phi_i, 0)$  of arriving at a phase space point  $\phi_f = (q_f, p_f)$  at time  $t_f = T$  having started from  $\phi_i = (q_i, p_i)$  at time  $t_i = 0$ ? The putative answer is

$$K(\phi_f, T|\phi_i, 0) = \int [\mathcal{D}\phi] \delta(\phi - \phi_{\text{cl}}). \quad (23)$$

Here,  $[\mathcal{D}\phi]$  means integration is to be carried out in the space of all paths from  $\phi_i$  to  $\phi_f$ . The extremal phase space path  $\phi_{\text{cl}}$  is related to the solution of the boundary-value problem for the Euler–Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \quad (24)$$

with the boundary conditions  $q(0) = q_i$  and  $q(T) = q_f$ . Because

$$\delta(\phi - \phi_{\text{cl}}) = \delta\left(\frac{\delta S}{\delta \phi}\right) \det\left[\frac{\delta^2 S}{\delta \phi(t') \delta \phi(t'')}\right], \quad (25)$$

one may further take the Fourier transform of the Dirac delta and exponentiate the determinant using Grassmannian ghost variables  $c$  and  $\bar{c}$  to yield

$$K(\phi_f, T|\phi_i, 0) = \int [\mathcal{D}\phi] \mathcal{D}\lambda \mathcal{D}\bar{c} \mathcal{D}c \exp\left(i\lambda \frac{\delta S}{\delta \phi} + \bar{c} \frac{\delta^2 S}{\delta \phi^2} c\right). \quad (26)$$

If we define two anticommuting partners of  $t$ ,  $\bar{\theta}$  and  $\theta$ , and assemble the variables  $\phi, \lambda, \bar{c}, c$  into a single combination

$$\Phi = \phi + \bar{\theta}\bar{c} + \theta c + i\bar{\theta}\theta\lambda, \quad (27)$$

then it is possible to rewrite (26) in a very compact and elegant supersymmetric form [5]:

$$K(Q_f, T|Q_i, 0) = \int [\mathcal{D}Q] \mathcal{D}P \exp\left(-\int d\bar{\theta} d\theta S[\Phi]\right), \quad (28)$$



which bears the formal similarity to the quantum path integral.

However, Eq. (23) is not always well defined. This is because the Cauchy problem for the Euler–Lagrange equations (24) with the initial conditions  $q(0) = q_i$  and  $\dot{q}(0) = \dot{q}_i$  is not equivalent to the boundary-value problem for these equations with the boundary conditions  $q(0) = q_i$  and  $q(T) = q_f$ . To illustrate, we refer to a harmonic oscillator  $L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$ . For  $q_i = q_f = 0$ , we have a one-parameter family of solutions  $q = \alpha \sin \omega t$ , labelled by  $\alpha$ . [This nonuniqueness has nothing to do with violating Laplace’s determinism because the solution to the Cauchy problem with arbitrary  $q_i = Q$  and  $\dot{q}_i = V$  is unique:  $q = Q \sin \omega t + (V/\omega) \cos \omega t$ .] Keeping in mind that  $K(\phi_f, T | \phi_i, 0)$  is the transition amplitude, we should assign a certain statistical weight  $\rho(\alpha)$  to every  $\phi_{\text{cl}}(\alpha)$ . The question now arises of what is the nature of  $\rho(\alpha)$ ? How can  $\rho(\alpha)$  be calculated?

It seems expedient to defer conclusive answer to this question<sup>5</sup> till future insights. For now, we attempt to remedy the situation by taking, as the starting point, the principle of least action in Jacobi’s form [10]. Recalling the analysis of a particle moving to the top of a hill, on condition that the total energy is zero, Eq. (2), one has formed the impression of this formulation as best suited to the description of indeterministic dynamics.

We now outline general features of Jacobi’s action<sup>6</sup>. Consider a nonrelativistic system described by a  $n$ -dimensional configuration space with coordinates  $q^a$ ,  $a = 1, 2, \dots, n$ . Let this system be moving along a path  $q^a(\sigma)$  whose argument  $\sigma$  ranges from 0 to 1. We denote  $q'^a = dq^a/d\sigma$ , and introduce the Newtonian metric  $m_{ab}(q)$  (for a single point particle of mass  $m$ , with the use of Cartesian coordinates,  $m_{ab} = m \delta_{ab}$ ). Jacobi’s action is an integral over the configuration space trajectory,

$$\bar{S} = \int_0^1 d\sigma \sqrt{m_{ab}(q) q'^a q'^b} \sqrt{2[E - U(q)]}, \quad (29)$$

where  $U(q)$  is the potential energy. In (29), the physical time interval between initial and final configurations is not fixed. By contrast, the total energy of the system  $E$  is fixed.

Varying (29) gives a trajectory  $q(\sigma)$ . With the knowledge of  $q(\sigma)$ , it is possible to determine how the system evolves in time using a supplementary condition

$$\frac{1}{2} m_{ab} \dot{q}^a \dot{q}^b + U(q) = E, \quad (30)$$

where  $\dot{q}^a = dq^a/dt$ .

Since Jacobi’s action (29) is invariant under the change of parametrization  $\sigma \rightarrow f(\sigma)$  with  $f(0) = 0$  and  $f(1) = 1$ , the Hamiltonian associated with the Lagrangian in (29) vanishes identically,

$$\mathcal{H} = q^a \frac{\partial L}{\partial q^a} - L = 0. \quad (31)$$

To put it differently, reparametrization invariance of the action (29) leads to a constraint

$$\mathcal{H}(q, p) = \frac{1}{2} m^{ab} p_a p_b + U(q) - E \approx 0, \quad (32)$$

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<sup>5</sup>One possibility is to abandon the “intuitively obvious” expression (23), and instead proceed directly from the “quantum-mechanically motivated” supersymmetric construction (28).

<sup>6</sup>Our brief review is loosely patterned on the detailed exposition of Ref. [11].

where  $p_a$  is the momentum conjugate to the configuration coordinate  $q^a$ ,

$$p_a = \frac{\partial L}{\partial q'^a} = \frac{q'_a}{\sqrt{m_{ab} q'^a q'^b}} \sqrt{2[E - U(q)]}. \quad (33)$$

Because the canonical Hamiltonian  $\mathcal{H}$  is zero, there are no secondary constraints, and  $\mathcal{H}$  is trivially first class. The action in canonical form is

$$\bar{S} = \int_0^1 d\sigma (p_a q'^a - N\mathcal{H}), \quad (34)$$

where  $N$  is a Lagrange multiplier, whose variation enforces the constraint (32).

The action (34) is to be varied with  $q(0) = q_i$  and  $q(1) = q_f$  held fixed. The equations of motion following from (34) are

$$q'^a = Np^a, \quad p'_a = -N \left( \frac{1}{2} p_b p_c \frac{\partial m^{bc}}{\partial q^a} + \frac{\partial U}{\partial q^a} \right), \quad \frac{1}{2} p_a p^a + U(q) - E = 0, \quad (35)$$

where  $p^a = m^{ab} p_b$ . Combining the first and third of these equations gives

$$N = \left[ \frac{q'^a q'_a}{2(E - U)} \right]^{\frac{1}{2}}. \quad (36)$$

By (30),

$$\frac{dt}{d\sigma} = \left[ \frac{q'^a q'_a}{2(E - U)} \right]^{\frac{1}{2}}. \quad (37)$$

Therefore,  $dt = N d\sigma$ , and so

$$T = \int_0^1 d\sigma N(\sigma). \quad (38)$$

This suggests that  $N$  is the lapse in physical time associated with an increment in the variable  $\sigma$  parametrizing the phase space trajectory. Note that this interpretation for  $N$  will be maintained for as long as the flow of  $t$  is correlated with increasing  $\sigma$ . This, however, is not the case for indeterministic regimes of evolution.

The action (34) is invariant under infinitesimal reparametrizations  $\delta\sigma = \epsilon(\sigma)$  with  $\epsilon(0) = \epsilon(1) = 0$  if one takes the transformation laws

$$\delta q = \epsilon q', \quad \delta p = \epsilon p', \quad \delta N = (\epsilon N)', \quad (39)$$

which are generated by the first class constraint (32).

If we express the momenta  $p$  in terms of the velocities  $q'$ , then (34) becomes

$$\bar{S} = \int_0^1 d\sigma \left[ \frac{m_{ab} q'^a q'^b}{2N} + N(E - U) \right]. \quad (40)$$

Integrating away  $N$  from (40), we return to the original Jacobi action (29).

With this restating the principle of least action, do we succeed in rendering the solution to the boundary-value problem unique? The answer is: no. To see this, we refer to a particle on a sphere which moves under a constant force directed from the north pole to the south one. Evidently the extremal path between two fixed points on the sphere is not unique. For example, if the particle is to move between the north and south poles, and  $E$  takes a fixed value, then one meridian is equally appropriate for this journey as the other.

Rather than systematically prosecute the subject, we focus on one-dimensional systems (which seem to be not a matter of concern). We first define an invariant path-integral measure for deterministic reparametrization-invariant systems. Following [12, 13], we integrate over the coset space of all functions  $\phi(\sigma)$  and one-dimensional metrics  $N(\sigma)$  modulo reparametrizations,

$$\frac{\mathcal{D}\phi(\sigma) \mathcal{D}N(\sigma)}{\mathcal{D}f(\sigma)}, \quad (41)$$

where  $\mathcal{D}N(\sigma)/\mathcal{D}f(\sigma)$  can be shown [12, 13] to reduce to a conventional Lebesgue measure  $dT$ , with  $T$  being the physical time interval given by (36). By applying these results to the procedure of Ref. [5], we recast (28) in the form

$$K(Q_f|Q_i) = \int_0^\infty dT \int [\mathcal{D}Q] \mathcal{D}P \exp \left[ - \int_0^1 d\sigma d\bar{\theta} d\theta (PQ' - T\mathcal{H}) \right], \quad (42)$$

where the conjugate supervariables  $Q$  and  $P$  are patterned after Eq. (27), and

$$\mathcal{H}(Q, P) = \frac{1}{2m} P^2 + U(Q) - E. \quad (43)$$

A modifications of the path integral for indeterministic systems can be ascertained by the example of a particle that moves to the top of the hill  $U(q) = -U_0 |q|$ , equilibrates at  $q = 0$  for an arbitrary period of time  $T$ , and then descends down the hill. The initial and final stages of this process, that is, the ascent and descent, are essentially deterministic. Hence, the classical transition amplitude for these stages is deduced from (42)–(43).

Care must be exercised in treating the indeterministic stage—that is, the stay at the top. Let us assume that  $T$  is a discrete variable taking values  $0, \ell, 2\ell, \dots$ . The prior probability that the particle will be at rest after completing one quantum of time  $\ell$  is  $\frac{1}{2}$ . After a lapse of two quanta of time  $2\ell$ , this quantity is  $(\frac{1}{2})^2$ . And so on. With this assumption, the  $T$ -integration is substituted for a discrete sum, and hence

$$K(\phi_f = 0 | \phi_i = 0) = \left( \frac{1}{2} + \frac{1}{4} + \dots \right) \int [\mathcal{D}q] [\mathcal{D}p] \delta(q) \delta(p) = \frac{1}{16}. \quad (44)$$

Here, the end-point integrals of the Dirac deltas over half-infinite intervals are understood as appropriate limits of integrals of sequences of functions, such as

$$\int_0^\infty dx \delta(x) \varphi(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{2} \int_{-\infty}^\infty dx \frac{\epsilon}{\pi(x^2 + \epsilon^2)} \varphi(x) = \frac{1}{2} \varphi(0). \quad (45)$$

It would be interesting to see if it is possible to bridge this random dynamics arising from spontaneous equilibrium breaking with that owing its origin to 't Hooft's information loss condition (for a discussion of this condition and further references see [14]).

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